ON THE MAXIMAL FUNCTION FOR THE GENERALIZED ORNSTEIN-UHLENBECK SEMIGROUP.

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Abstract. In this note we consider the maximal function for the generalized Ornstein-Uhlenbeck semigroup in $\mathbb{R}$ associated with the generalized Hermite polynomials $\{H_n^\mu\}$ and prove that it is weak type (1,1) with respect to $d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx$, for $\mu > -1/2$ as well as bounded on $L^p(d\lambda_\mu)$ for $p > 1$.

1. Introduction and Preliminaries

The generalized Hermite polynomials were defined by G. Szëgo in [14] (see problem 25, pag 380) as being orthogonal polynomials with respect to the measure $d\lambda(x) = d\lambda_\mu(x) = |x|^{2\mu} e^{-|x|^2} dx$, with $\mu > -1/2$. In his doctoral thesis T. S. Chihara [2] (see also [3]) studied them in detail. In this paper we consider the definition of the generalized Hermite polynonials given by M. Rosenblum in [10].

Let us denote by H_n^μ this generalized Hermite polynomial of degree n, then for n even

$$H_{2m}^\mu(x) = (-1)^m (2m)! \frac{\Gamma(\mu + \frac{1}{2})}{\Gamma(m + \mu + \frac{1}{2})} L_m^{\mu - \frac{1}{2}}(x^2) \tag{1.1}$$

and for n odd

$$H_{2m+1}^\mu(x) = (-1)^m (2m+1)! \frac{\Gamma(\mu + \frac{3}{2})}{\Gamma(m + \mu + \frac{3}{2})} x L_m^{\mu + \frac{1}{2}}(x^2), \tag{1.2}$$

L_m^γ being the γ-Laguerre polynomial of degree m.

Thus, for every $n \in \mathbb{N}$,

$$\|H_n^\mu\|_{L^2(d\lambda)} = \left(\frac{2^n (n!)^2 \Gamma(\mu + 1/2)}{\gamma_\mu(n)} \right)^{1/2},$$

where $\gamma_\mu(m)$ is a generalized factorial defined by,

$$\gamma_\mu(2m) = \frac{2^{2m} m! \Gamma(m + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m)! \frac{\Gamma(m + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})},$$

$$\gamma_\mu(2m+1) = \frac{2^{2m+1} m! \Gamma(m + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} = (2m)! \frac{\Gamma(m + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})} \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{3}{2})}.$$

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The generalized Hermite polynomials $\{H_n^\mu\}$ have a generating function (2.5.8) of [10]) which involves the generalized exponential function e_μ defined by

$$e_\mu(z) = \sum_{m=0}^{\infty} \frac{z^m}{\gamma_\mu(m)}. \tag{1.3}$$

On the other hand each generalized Hermite polynomial satisfies the following differential equation, see [3],

$$(H_n^\mu)''(x) + 2(\frac{\mu}{x} - x)(H_n^\mu)'(x) + 2(n - \mu\frac{\theta_n}{x^2})H_n^\mu(x) = 0, \tag{1.4}$$

with

$$\theta_n = \begin{cases} 1 & \text{if} \quad n \text{ is odd,} \\ 0 & \text{if} \quad n \text{ is even.} \end{cases}$$

and $n \geq 0$.
Therefore, by considering the (differential-diference) operator

$$L_\mu = \frac{1}{2}\frac{d^2}{dx^2} + (\frac{\mu}{x} - x)\frac{d}{dx} - \mu\frac{I - \tilde{I}}{2x^2}, \tag{1.5}$$

where $If(x) = f(x)$ and $\tilde{I}f(x) = f(-x)$, H_n^μ turns out to be an eigenfunction of L_μ with eigenvalue $-n$.

Now we can define a Markov semigroup, see D. Bakry [1], by

$$P_t(x, dy) = \sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n(n!)^2} H_n^\mu(x)H_n^\mu(y)e^{-nt}\lambda(dy). \tag{1.6}$$

This semigroup is entirely characterized by the action on positive or bounded measurable functions by

$$T_\mu^t f(x) = \int_{-\infty}^{\infty} f(y)P_t(x, dy).$$

Thus the family of operators $\{T_\mu^t\}_{t\geq 0}$ is then a conservative semigroup of operators with generator L_μ, that we will call the generalized Ornstein-Uhlenbeck semigroup. Therefore,

$$\frac{\partial T_\mu^t f(x)}{\partial t} = L_\mu T_\mu^t f(x).$$

For $\mu = 0$, $\{T_\mu^t\}$ reduces to the Ornstein-Uhlenbeck semigroup whose behavior on L^p was studied by B. Muckenhoupt in [7] for the one-dimensional case. By using the generalized Mehler's formula (2.6.8) of [10]: for x, $y \in \mathbb{R}$ and $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{\gamma_\mu(n)}{2^n(n!)^2} H_n^\mu(x)H_n^\mu(y)z^n = \frac{1}{(1 - z^2)^{\mu+1/2}} e^{-\frac{z^2(x^2+y^2)}{1-z^2}} e_\mu\left(\frac{2xyz}{1 - z^2}\right). \tag{1.7}$$

we can obtain the following integral expression of this generalized Ornstein-Uhlenbeck semigroup $\{T_\mu^t\}$,

$$T_\mu^t f(x) = \frac{1}{(1 - e^{-2t})^{\mu+1/2}} \int_{-\infty}^{\infty} e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_\mu\left(\frac{2xye^{-t}}{1 - e^{-2t}}\right) f(y)|y|^{2\mu}e^{-|y|^2}dy. \tag{1.8}$$

In the following section we will consider the maximal operator associated with $\{T_\mu^t\}_{t>0}$, and prove it is weak type $(1, 1)$ with respect to the measure λ, bounded in L^∞ and therefore L^p bounded for $1 < p < \infty$ with respect to λ. It is important to

observe that since $\{T_\mu^t\}_{t>0}$ is not a convolution semigroup, its associated maximal operator is not bounded by the Hardy-Littlewood maximal operator and therefore in order to prove the weak $(1,1)$ inequality with respect to λ it is needed to develop new techniques. The case $\mu = 0$, that as we already said corresponds to the maximal operator of the Ornstein-Uhlenbeck semigroup, was proved by Sjögren in [11] in any dimension.
We will use repeatedly that

$$|x|^k e^{-x^2} \leq C e^{-x^2/2} \leq C, \quad \forall\, x \in \mathbb{R}. \tag{1.9}$$

The constant C which will appear throughout this paper may be different on each occurrence.

2. The maximal function of the generalized Ornstein Uhlenbeck semigroup.

Let us define the generalized Ornstein-Uhlenbeck maximal function as

$$T_\mu^* f(x) \quad = \quad \sup_{t>0} |T_\mu^t f(x)|, \tag{2.1}$$

for each $x \in \mathbb{R}$. Taking $r = e^{-t}$, we can write

$$T_\mu^* f(x) = \sup_{0<r<1} \left| \int_{-\infty}^{\infty} K_r(x,y) f(y)\, d\lambda(y) \right|,$$

with

$$K_r(x,y) = \frac{1}{\Gamma(\mu+\frac{1}{2})(1-r^2)^{\mu+\frac{1}{2}}} e^{-(x^2+y^2)\frac{r^2}{1-r^2}} e_\mu\Big(\frac{2xyr}{1-r^2}\Big).$$

The main result of this paper is summarized in

Theorem 2.1. *For $\mu > -1/2$,*

i) *T_μ^* is weak type $(1,1)$ with respect to λ, i.e. there exists a real constant $C > 0$ such that for every $\eta > 0$*

$$\lambda\{x \in \mathbb{R} : T_\mu^* f(x) > \eta\} \leq \frac{C}{\eta} \|f\|_{1,\lambda}, \tag{2.2}$$

where $\|f\|_{1,\lambda} = \int_{\mathbb{R}} |f(y)| d\lambda(y)$.

ii) *T_μ^* is bounded in L^∞, i. e. there exists a real constant $C > 0$ such that*

$$||T_\mu^* f||_\infty \leq C ||f||_\infty \tag{2.3}$$

where $\|f\|_\infty$ represents the L^∞ norm.

Corollary 2.2. *For $\mu > -1/2$ and $p > 1$,*

$$\|T_\mu^* f\|_{p,\lambda} \leq C\, \|f\|_{p,\lambda}, \tag{2.4}$$

where $\|f\|_{p,\lambda}^p = \int_{\mathbb{R}} |f(y)|^p d\lambda(y)$.

This corollary follows from Marcinkiewicz interpolation theorem between the weak type $(1,1)$ and the boundedness in L^∞ which will be proved in Theorem 2.1. In order to prove Theorem 2.1 we will introduce well known bounds for the functions e_μ and prove two propositions. The first one due to I. P. Natanson and B. Muckenhoupt ([8] and [7]) is a sort of a generalized Young's inequality for Borel measures, that we will write it only for the particular case of the measure λ and the other

one has to do with the biggest function whose density distribution as a function of η with respect to λ is bounded by C/η.

Properties of e_μ

It can be proved, see (2.2.3) of [10], that the generalized exponential function e_μ can be written as,

$$e_\mu(x) = \Gamma(\mu + 1/2)(2/x)^{\mu-1/2}(I_{\mu-1/2}(x) + I_{\mu+1/2}(x)),$$

where I_ν denotes the modified Bessel function. Then, according to [15, (2), p. 77, and (2), p. 203], we have the following estimates that will be useful in the sequel

$$|e_\mu(x)| \le e_\mu(|x|) \le C(1 + |x|)^{-\mu} e^{|x|}, \quad x \in \mathbb{R}. \tag{2.5}$$

Also, e_μ admits the following integral representations depending on the values of μ [10],

(1) if $\mu > 0$ then

$$e_\mu(x) = \frac{1}{B(\frac{1}{2}, \mu)} \int_{-1}^{1} e^{xt}(1 - t)^{\mu-1}(1 + t)^{\mu}\, dt, \tag{2.6}$$

(2) if $\mu = 0$ then

$$e_0(x) = e^x, \tag{2.7}$$

(3) if $-\frac{1}{2} < \mu < 0$ then

$$e_\mu(x) = e^x + \frac{\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu + 1)} \int_{-1}^{1} (e^{xt} - e^x)(1 - t)^{\mu-1}(1 + t)^{\mu} dt \tag{2.8}$$

According to (2.6) it is clear that $e_\mu(x) \ge 0$, for $\mu \ge 0$, $x \in \mathbb{R}$. However, this one is not the case when $-1/2 < \mu < 0$. Indeed, assume that $-1/2 < \mu < 0$. Since $e^u - 1 \ge u$, $u > 0$, we can write

$$\begin{aligned} e^{-x} e_\mu(x) &= 1 + \frac{\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu + 1)} \int_{-1}^{1} (e^{x(t-1)} - 1)(1 - t)^{\mu-1}(1 + t)^{\mu} dt \\ &\le 1 - \frac{x\mu}{\mu + 1/2} \frac{1}{B(1/2, \mu + 1)} \int_{-1}^{1} (1 - t)^{\mu}(1 + t)^{\mu} dt, \quad x < 0. \end{aligned}$$

Hence, there exists $x_0 > 0$ such that $e_\mu(x) < 0$ for every $x < -x_0$.

From the above we infer that the generalized Ornstein-Uhlenbeck semigroup $\left\{T_\mu^t\right\}_{t>0}$ is a positive one when $\mu \ge 0$ but it is not when $-1/2 < \mu < 0$.

Proposition 2.3. *(Natanson) Let f and g be two $L^1(d\lambda)$ functions. Let us assume that g(y) is nonnegative and there is an $x \in \mathbb{R}$ such that $g(y)$ is monotonically increasing for $y \le x$ and monotonically decreasing for $x \le y$, then*

$$\left| \int g(y) f(y)\, d\lambda(y) \right| \le \|g\|_{1,\lambda} \mathcal{M}_\lambda f(x) \tag{2.9}$$

where

$$\mathcal{M}_\lambda f(x) = \sup_{x \in I} \frac{1}{\lambda(I)} \int_I |f(y)|\, d\lambda(y)$$

is the Hardy-Littlewood maximal fuction of f with respect to λ. Moreover the Hardy-Littlewood maximal fuction $\mathcal{M}_\lambda f$ is weak type (1,1) and strong type (p,p) for $p > 1$ with respect to the measure λ.

A proof of this proposition can be found in [7].

Proposition 2.4. *For $\mu > -1/2$, there is a real constant $C > 0$ such that the distribution function with respect to λ of the function*

$$h(x) = \max\left(\frac{1}{|x|}, |x|\right)\frac{e^{x^2}}{|x|^{2\mu}}$$

satisfies the inequality

$$\lambda\{x \in \mathbb{R} : h(x) > \eta\} \leq \frac{C}{\eta},$$

for any $\eta > 0$.

Proof. Since λ is a finite measure, it is enough to prove this result for $\eta \geq e$. Besides, due to the fact that h is even and λ is symmetric, then $\lambda\{x \in \mathbb{R} : h(x) > \eta\} = 2\lambda\{x > 0 : h(x) > \eta\}$. Now

$$\begin{aligned}
\lambda\{x > 0 : h(x) > \eta\} &\leq \lambda\left\{0 < x < 1 : \frac{1}{x^{2\mu+1}} > \eta/e\right\} \\
&\quad +\lambda\left\{x > 1 : \frac{e^{x^2}}{x^{2\mu-1}} > \eta\right\} \\
&= \int_0^{(e/\eta)^{\frac{1}{2\mu+1}}} x^{2\mu}e^{-x^2}dx \\
&\quad + \int_{x_0}^{\infty} x^{2\mu}e^{-x^2}dx \\
&= I + II
\end{aligned}$$

with $x_0 > 1$ and $\frac{e^{x_0^2}}{x_0^{2\mu-1}} = \eta$. Let us observe that

$$I \leq \int_0^{(e/\eta)^{1/(2\mu+1)}} x^{2\mu}dx = \frac{e}{(1+2\mu)\eta},$$

and

$$II \leq Cx_0^{2\mu-1}e^{-x_0^2} = \frac{C}{\eta}.$$

For last inequality see [5]. From these two bounds the conclusion of this proposition follows. □

Proof. of Theorem 2.1.

In order to prove this theorem it suffices to show that there exists $C > 0$ such that

$$\lambda\{x \in (0,\infty) : T^*_{\mu,+}f(x) > \eta\} \leq \frac{C}{\eta}\|f\|_{1,\lambda}, \;\; \eta > 0, \tag{2.10}$$

and

$$\|T^*_{\mu,+}f\|_\infty \leq C\|f\|_\infty \tag{2.11}$$

for every $f \geq 0$, where

$$T^*_{\mu,+}f(x) = \sup_{t>0}|T^\mu_{t,+}f(x)|,$$

and

$$T^t_{\mu,+}f(x) = \frac{1}{(1-e^{-2t})^{\mu+1/2}}\int_0^\infty e^{-\frac{e^{-2t}(x^2+y^2)}{1-e^{-2t}}} e_\mu\left(\frac{2xye^{-t}}{1-e^{-2t}}\right) f(y)|y|^{2\mu}e^{-|y|^2}dy.$$

Indeed, let us write $r = e^{-t}$, with $t > 0$. By (2.5), we have that

$$K_r(x, y) \leq K_r(|x|, |y|), \;\; x, y \in \mathbb{R}.$$

Then

$$|T^t_\mu f(x)| \leq T^t_{\mu,+}|f|(|x|) + T^t_{\mu,+}|\tilde{f}|(|x|), \;\; x \in \mathbb{R},$$

being $\tilde{f}(x) = f(-x)$, $x \in \mathbb{R}$. Hence,

$$T^*_\mu f(x) \leq T^*_{\mu,+}|f|(|x|) + T^*_{\mu,+}|\tilde{f}|(|x|), \;\; x \in \mathbb{R},$$

and we can write, for every $\eta > 0$,

$$\begin{aligned}
\lambda\{x \in \mathbb{R} : T^*_\mu f(x) > \eta\} &\leq \lambda\{x \in \mathbb{R} : T^*_{\mu,+}|f|(|x|) > \eta/2\} \\
&\quad +\lambda\{x \in \mathbb{R} : T^*_{\mu,+}|\tilde{f}|(|x|) > \eta/2\} \\
&\leq 2(\lambda\{x \in (0,\infty) : T^*_{\mu,+}|f|(x) > \eta/2\} \\
&\quad +\lambda\{x \in (0,\infty) : T^*_{\mu,+}|\tilde{f}|(x) > \eta/2\}).
\end{aligned}$$

Thus (2.2) follows from (2.10), (2.11) and the fact that $\|f\|_{1,\lambda} = \|\tilde{f}\|_{1,\lambda}$ and $\|f\|_\infty = \|\tilde{f}\|_\infty$.

From now on let us assume $f \geq 0$ and $x > 0$. First let us prove the weak type $(1, 1)$ inequality.

(1) Case $\mu = 0$. This case corresponds to the Ornstein-Uhlenbeck maximal operator which was proved to be weak type $(1, 1)$ by B. Muckenhoupt in [7].

(2) Case $\mu > -1/2$. By using (2.5) we can write

$$\begin{aligned}
T^t_{\mu,+}f(x) &\leq \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{(x^2+y^2)r^2}{1-r^2}+\frac{2xyr}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y) \\
&= \frac{Ce^{x^2}}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{|x-ry|^2}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y) \\
&= \frac{Ce^{x^2}}{(1-r^2)^{\mu+1/2}}\left(\int_0^{x/2r}+\int_{x/2r}^{4x/r}+\int_{4x/r}^\infty\right)e^{-\frac{|x-ry|^2}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y) \\
&= C(K_{1,r}f(x) + K_{2,r}f(x) + K_{3,r}f(x)).
\end{aligned}$$

Let us observe that if $0 < y < x/2r$, then $x - ry > x/2$ and

$$\frac{1}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} \leq \frac{1}{(1-r^2)^{\mu+1/2}} + \frac{x^{-2\mu}}{(1-r^2)^{1/2}},$$

thus

$$K_{1,r}f(x) \leq Ce^{x^2}\left(\frac{1}{(1-r^2)^{\mu+1/2}} + \frac{x^{-2\mu}}{(1-r^2)^{1/2}}\right)e^{-\frac{x^2}{4(1-r^2)}}\|f\|_{1,\lambda} \leq C\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda},$$

where last inequality is obtained as an application of (1.9).

On the other hand, if $y > \frac{4x}{r}$, then $ry - x > x$, and again by applying (1.9) repeatedly in the sequel below

$$\begin{aligned}
\frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} &= \frac{e^{-\frac{|x-ry|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2x(ry-x)+2x^2}{1-r^2}\right)^{-\mu} \\
&\leq C\, e^{-\frac{x^2}{2(1-r^2)}}\left(\frac{1}{(1-r^2)^{\mu+1/2}}+\frac{x^{-\mu}}{(1-r^2)^{\frac{\mu+1}{2}}}+\right. \\
&\qquad\left.\frac{x^{-2\mu}}{(1-r^2)^{1/2}}\right) \\
&\leq \frac{C}{x^{2\mu+1}},
\end{aligned}$$

we get

$$K_{3,r}f(x) \leq C\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda}.$$

Finally for $\frac{x}{2r} \leq y \leq \frac{4x}{r}$ we have the following estimate

$$\frac{1}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2rxy}{1-r^2}\right)^{-\mu} \leq \frac{1}{x^{2\mu+1}}+\frac{x^{-2\mu}}{(1-r^2)^{1/2}}, \tag{2.12}$$

which is immediate for $\mu \geq 0$ and for $\mu < 0$ one has to argue between $\frac{2rxy}{1-r^2} \leq 1$ and its complement. Now by taking into account inequality (2.12) we are ready to estimate $K_{2,r}f(x)$ and for that we consider two cases. If $0 < r \leq 1/2$ we have

$$K_{2,r}f(x) \leq C\left(\frac{1}{x}+1\right)\frac{e^{x^2}}{x^{2\mu}}\|f\|_{1,\lambda},$$

and, if $1/2 < r < 1$ then

$$K_{2,r}f(x) \leq C\left(\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda}+\frac{e^{x^2}}{(1-r^2)^{1/2}x^{2\mu}}\int_0^\infty N(r,x,y)f(y)d\lambda(x)\right),$$

with

$$N(r,x,y) = \begin{cases} 1 & \text{if} \quad y \in \left[x, \frac{x}{r}\right] \\ e^{-\frac{|x-ry|^2}{1-r^2}} & \text{if} \quad y \in \left[\frac{x}{2r}, \frac{4x}{r}\right] \setminus \left[x, \frac{x}{r}\right] \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Since $N(r,x,.)$ is a Natanson kernel (see (2.9)), we get

$$K_{2,r}f(x) \leq C\left(\frac{e^{x^2}}{x^{2\mu+1}}\|f\|_{1,\lambda}+\frac{e^{x^2}}{x^{2\mu}(1-r^2)^{1/2}}\|N(r,x,.)\|_{1,\lambda}\,\mathcal{M}_\lambda f(x)\right).$$

Let us prove that

$$\|N(r,x,.)\|_{1,\lambda} \leq Cx^{2\mu}(1-r^2)^{1/2}e^{-x^2}. \tag{2.14}$$

Indeed,

$$
\begin{aligned}
\int_{\mathbb{R}} N(r,x,y)\,d\lambda(y) &= \int_x^{x/r} e^{-y^2} y^{2\mu}\,dy + \int_{x/2r}^{x} e^{-\frac{|x-ry|^2}{1-r^2}} e^{-y^2} y^{2\mu}\,dy \\
&\quad + \int_{x/r}^{4x/r} e^{-\frac{|x-ry|^2}{1-r^2}} e^{-y^2} y^{2\mu}\,dy \\
&\sim x^{2\mu}\left(\int_x^{x/r} e^{-y^2}\,dy + e^{-x^2}\int_{x/2r}^{x} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right. \\
&\quad \left. + e^{-x^2}\int_{x/r}^{4x/r} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right) \\
&\leq C\,x^{2\mu}e^{-x^2}\left(\min\left(\frac{1}{x},(1-r)x\right)\right. \\
&\quad \left. + \int_{\mathbb{R}} e^{-\frac{|rx-y|^2}{1-r^2}}\,dy\right) \\
&\leq C\,x^{2\mu}(1-r^2)^{1/2}e^{-x^2}.
\end{aligned}
$$

Now gathering together all the bounds obtained above, we get

$$T^t_{\mu,+}f(x) \leq C(h(x)\|f\|_{1,\lambda} + \mathcal{M}_\lambda f(x)),$$

for all $t > 0$, where h is the function defined in Proposition 2.4. Thus the weak type $(1,1)$ of $T^*_{\mu,+}$ follows from propositions 2.3 and 2.4.

Now let us take care of the boundedness of $T^*_{\mu,+}$ in L^∞.
For the case $\mu \geq 0$ this boundedness is immediate since its kernel is non-negative and its integral equals 1. Therefore let us study just the case $-1/2 < \mu < 0$. By using (2.5) and proceeding like in case 2 of the weak type $(1,1)$ inequality

$$
\begin{aligned}
T^*_{\mu,+}f(x) &\leq \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{(x^2+y^2)r^2}{1-r^2}+\frac{2xyr}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} f(y)\,d\lambda(y) \\
&\leq \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{|rx-y|^2}{1-r^2}}\left(1+\frac{2xyr}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy\;\|f\|_\infty \\
&= \frac{C}{(1-r^2)^{\mu+1/2}}\int_0^\infty e^{-\frac{|rx-y|^2}{1-r^2}}\left(1+\frac{2(rx-y)y}{1-r^2}+\frac{2y^2}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy\;\|f\|_\infty \\
&\leq C\Bigg(\int_0^\infty \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{\mu+1/2}}\left(1+\frac{2|rx-y|y}{1-r^2}\right)^{-\mu} y^{2\mu}\,dy \\
&\quad + \int_0^\infty \frac{e^{-\frac{|rx-y|^2}{1-r^2}}}{(1-r^2)^{1/2}}\,dy\Bigg)\|f\|_\infty
\end{aligned}
$$

In order to prove that the first integral of last inequality is bounded by a constant independent of r, y, and x first we use (1.9) to get the inequality

$$\left(\frac{2|rx-y|y}{1-r^2}\right)^{-\mu} e^{-\frac{|rx-y|^2}{1-r^2}} \leq C\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu} e^{-\frac{|rx-y|^2}{2(1-r^2)}},$$

then we split the integral in two subintervals one from 0 to $\sqrt{1-r^2}$ and the other from $\sqrt{1-r^2}$ to ∞ and we call them I and II. Now we proceed to bound each

part.

$$\begin{aligned} I &= \int_0^{\sqrt{1-r^2}} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}} \left(1+\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu}\, dy \\ &\leq \int_0^{\sqrt{1-r^2}} \frac{y^{2\mu}}{(1-r^2)^{\mu+1/2}}\, dy + \int_0^{\sqrt{1-r^2}} \frac{y^{\mu}}{(1-r^2)^{(\mu+1)/2}}\, dy \leq C, \end{aligned}$$

and

$$\begin{aligned} II &= \int_{\sqrt{1-r^2}}^{\infty} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}} \left(1+\left(\frac{y}{(1-r^2)^{1/2}}\right)^{-\mu}\right) y^{2\mu}\, dy \\ &\leq \int_{\sqrt{1-r^2}}^{\infty} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}} (\sqrt{1-r^2})^{2\mu}\, dy + \int_{\sqrt{1-r^2}}^{\infty} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{\mu+1/2}} \frac{y^{\mu}}{(1-r^2)^{-\mu/2}}\, dy \\ &\leq 2\int_0^{\infty} \frac{e^{-\frac{|rx-y|^2}{2(1-r^2)}}}{(1-r^2)^{1/2}} dy \leq C. \end{aligned}$$

This ends the proof of the boundedness of $T^*_{\mu,+}$ in L^∞ and at the same time the proof of Theorem 2.1.

□

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